EXISTENCE AND MULTIPLICITY OF A NONHOMOGENEOUS POLYHARMONIC EQUATION WITH CRITICAL EXPONENTIAL GROWTH IN EVEN DIMENSION

ABHISHEK SARKAR

ABSTRACT. In this paper we study the existence of at least two positive weak solutions for an inhomogeneous fourth order equation with Navier boundary data involving nonlinearities of critical growth with a bifurcation parameter λ in $\mathbb{R}^{2m}$. We establish here the lower and upper bound for λ which determine multiplicity and non-existence respectively.

1. Introduction

Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded domain. In this context we study the existence of multiple solutions in $W^{m,2}_{\mathcal{N}}(\Omega) = \{u \in W^{m,2}(\Omega) : \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } 0 \leq j < \frac{m}{2}\}$ of the following $2m$-th order problem with Navier boundary condition

$$(P) \qquad \begin{cases} \left.\begin{aligned} (-\Delta)^m u &= \mu u|u|^p e^{u^2} + \lambda h(x) \\ u &> 0 \end{aligned}\right\} & \text{in } \Omega, \\ u = \Delta u = 0 = .. = \Delta^{m-1} u & \text{on } \partial\Omega \end{cases}$$

where $h \geq 0$ in Ω, $\|h\|_{L^2(\Omega)} = 1$, $\lambda > 0, \mu = 1$ if $p > 0$ and $\mu \in (0, \lambda_1(\Omega))$ if $p = 0$. Also assume that $\lambda_1(\Omega)$ be the first eigenvalue of $(-\Delta)^m$ on $W^{m,2}_{\mathcal{N}}(\Omega)$ with Navier boundary condition and which is strictly positive. The existence of multiple solutions for analogous problems in higher dimension with critical exponent have been studied in [5], [2] for the Dirichlet boundary condition and in [11] for Navier boundary condition. The corresponding problem for second order elliptic equations have been studied in [8] for dimension two, and in [9] for higher dimensions. The critical growth for the nonlinearity is $u \mapsto |u|^p u$, $p = \frac{4m}{n-2m}$, when $n \geq 2m+1$ from the Sobolev embedding in $\mathbb{R}^n$. In [7] Moser proved the following,

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a bounded domain. There exists a constant $C_n > 0$ such that for any $u \in W^{1,n}_0(\Omega)$, $n \geq 2$ with $\|\nabla u\|_{L^n(\Omega)} \leq 1$, then*

$$\int_\Omega e^{\alpha|u|^p} dx \leq C_n |\Omega|, \tag{1.1}$$

where

$$p = \frac{n}{n-1}, \ \alpha_n := n w_{n-1}^{\frac{1}{n-1}},$$

and w_{n-1} is the surface measure of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Furthermore the integral on the left hand side can be made arbitrarily large if $\alpha > \alpha_n$ by appropriate choice of u with $\|\nabla u\|_{L^n(\Omega)} \leq 1$. The embedding

$$W^{1,n}_0(\Omega) \ni u \mapsto e^{\alpha|u|^{\frac{n}{n-1}}} \in L^1(\Omega)$$

is compact for $\alpha < \alpha_n$ and is not compact for $\alpha = \alpha_n$.

In [1] Adams extended the above result of Moser to higher order Sobolev spaces. To state the result of Adams we define the following m-th order derivatives of $u \in C^m(\Omega)$:

$$\nabla^m u = \begin{cases} \Delta^{\frac{m}{2}} u & \text{for } m \text{ even}, \\ \nabla \Delta^{\frac{m-1}{2}} u & \text{for } m \text{ odd}. \end{cases}$$

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Furthermore, $\|\nabla^m u\|_p$ is the L^p norm of the function $|\nabla^m u|$, the usual Euclidean length of the vector $\nabla^m u$. We also denote $W_0^{m,\frac{n}{m}}(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ under the Sobolev norm

$$\|u\|_{W^{m,\frac{n}{m}}(\Omega)} = \left(\|u\|_{\frac{n}{m}}^{\frac{n}{m}} + \sum_{|\alpha|=1}^{m} \|D^\alpha u\|_{\frac{n}{m}}^{\frac{n}{m}}\right)^{\frac{m}{n}}. \tag{1.2}$$

Adams proved the following embedding:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If m is a positive integer less than n, then there exists a constant $C_0 = C(n,m) > 0$ such that for any $u \in W_0^{m,\frac{n}{m}}(\Omega)$ with $\|\nabla^m u\|_{\frac{n}{m}} \le 1$, then*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta |u(x)|^{\frac{n}{n-m}})dx \le C_0, \tag{1.3}$$

for all $\beta \le \beta_{n,m}$ where

$$\beta_{n,m} = \begin{cases} \frac{n}{w_{n-1}}\left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd,} \\ \frac{n}{w_{n-1}}\left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}} & \text{when } m \text{ is even.} \end{cases}$$

Furthermore, for any $\beta > \beta_{n,m}$, the integral can be made as large as possible by appropriate choice of u with $\|\nabla^m u\|_{\frac{n}{m}} \le 1$.

Remark 1.1. *We remark that for the case $n = 2m = 4$, Lu-Yang in [6] and in general Zhao-Chang [12] showed the existence of an explicit sequence for $n = 2m$ to prove the sharpness of the constant in $W_0^{m,\frac{n}{m}}(\Omega)$.*

Let $W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega)$ denote the following subspace of $W^{m,\frac{n}{m}}(\Omega)$:

$$W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega) = \left\{u \in W^{m,\frac{n}{m}}(\Omega) : \Delta^j u = 0, \text{ on } \partial\Omega \text{ for } 0 \le j \le [(m-1)/2]\right\}.$$

Note that $W_0^{m,\frac{n}{m}}(\Omega)$ is strictly contained in $W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega)$. Therefore,

$$\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega), \|\nabla^m u\|_{\frac{n}{m}} \le 1} \int_\Omega exp(\beta_{n,m}|u|^{\frac{n}{n-m}})dx \le \sup_{u \in W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega), \|\nabla^m u\|_{\frac{n}{m}} \le 1} \int_\Omega exp(\beta_{n,m}|u|^{\frac{n}{n-m}})dx.$$

Tarsi [10] later extended Adams' result for the larger space $W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega)$. The key step in her work is to embed $W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega)$ into a Zygmund space. We state her embedding theorem below

Theorem 1.3. *Let $n > 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there is a constant $C_n > 0$ such that for all $u \in W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega)$ with $\|\nabla^m u\|_{\frac{n}{m}} \le 1$, we have*

$$\int_\Omega e^{\beta|u|^{\frac{n}{n-m}}}dx < C_n|\Omega| \qquad \forall \beta \le \beta_{n,m} \tag{1.4}$$

and the constant $\beta_{n,m}$ appearing in (1.4) is sharp and $\beta_{n,m}$ is same as in Theorem 1.2

Remark 1.2. *Here we remark that the bilinear form*

$$(u,v) \mapsto \int_\Omega \nabla^m u \cdot \nabla^m v = \begin{cases} \int_\Omega \Delta^k u \Delta^k v & \text{if } m = 2k, \\ \int_\Omega \nabla(\Delta^k u)\cdot\nabla(\Delta^k v) & \text{if } m = 2k+1, \end{cases} \tag{1.5}$$

defines a scalar product on $W_0^{m,2}(\Omega)$ and $W_{\mathcal{N}}^{m,2}(\Omega)$. Furthermore if Ω is bounded this scalar product induces a norm equivalent to (1.2).

Therefore the above results imply that the problem (P) nonlinearity of critical growth.

Theorem 1.1. *There exist positive real numbers $\lambda_* \le \lambda^*$ with λ_* independent of h such that the problem (P) has at least two positive solutions for all $\lambda \in (0,\lambda_*)$ and no solution for all $\lambda > \lambda^*$.*

In spite of possible failure of Palais-Smale condition due to the presence of critical exponent we adapt the method of [9] to prove the existence of the first solution by a decomposition of Nehari manifold into three parts. However for the existence of second solution we rely on the refined version of the Mountain-Pass Lemma introduced by Ghoussoub-Preiss [3].

2. Decomposition of Nehari Manifold

Let $f(u) = \mu|u|^p u e^{u^2}$. The corresponding energy functional to the problem (P) is given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla^m u|^2 - \int_\Omega F(u) - \lambda \int_\Omega hu \tag{2.1}$$

where $F(u) = \int_0^u f(s)ds$. As the energy functional is not bounded below on $W^{m,2}_{\mathcal{N}}(\Omega)$, we need to study $J(u)$ on the Nehari manifold

$$\mathcal{M} = \{u \in W^{m,2}_{\mathcal{N}}(\Omega) \setminus \{0\} : \langle J'(u), u\rangle = 0\}, \tag{2.2}$$

where $J'(u)$ denotes the Frechet derivative of J at u, and $\langle .,.\rangle$ is the inner product. Here we note that $\mathcal{M}$ contains every nonzero solution of the problem (P). We note that for any $u \in W^{m,2}_{\mathcal{N}}(\Omega)$,

$$\langle J'(u), u\rangle = \int_\Omega |\nabla^m u|^2 - \int_\Omega f(u)u - \lambda \int_\Omega hu,$$
$$\langle J''(u)u, u\rangle = \int_\Omega |\nabla^m u|^2 - \int_\Omega f'(u)u^2.$$

Similarly to the method used in [9], We split $\mathcal{M}$ into three parts:

$$\mathcal{M}^0 = \{u \in \mathcal{M} : \langle J''(u)u, u\rangle = 0\},$$
$$\mathcal{M}^+ = \{u \in \mathcal{M} : \langle J''(u)u, u\rangle > 0\},$$
$$\mathcal{M}^- = \{u \in \mathcal{M} : \langle J''(u)u, u\rangle < 0\}.$$

3. Topological Properties of $\mathcal{M}^0, \mathcal{M}^+, \mathcal{M}^-$

Our first aim is to show, for some small λ, $\mathcal{M}^0 = \{0\}$. For this let $\zeta > 0$, if $p > 0$ and $\zeta < \frac{\lambda_1 - \mu}{\mu}$ if $p = 0$. Let $\Lambda = \{u \in W^{m,2}_{\mathcal{N}}(\Omega) : \int_\Omega |\nabla^m u|^2 \le (1+\zeta)\int_\Omega f'(u)u^2\}$. Lemma 3.2 implies that $\Lambda \ne \{0\}$. We now assume the following important hypothesis:

$$\lambda > 0, \|h\|_{L^2(\Omega)} = 1, \text{ and}$$
$$\inf_{u\in\Lambda\setminus\{0\}} \left(\mu \int_\Omega (p+2u^2)|u|^{p+2}e^{u^2} - \lambda \int_\Omega hu\right) > 0. \tag{3.1}$$

The condition (3.1) forces λ to be suitably small. Indeed we can prove the following.

Proposition 3.1. *Let*

$$\lambda < \mu C_0^{\frac{p+3}{p+4}} |\Omega|^{-\left(\frac{p+2}{2p+8}\right)} \tag{3.2}$$

where $C_0 = \inf_{u\in\Lambda\setminus\{0\}} \int_\Omega (p+2u^2)|u|^{p+2}e^{u^2} > 0$. *Then (3.1) holds.*

Proof. **Step 1:** $\inf_{u\in\Lambda\setminus\{0\}} \|u\|_{W^{m,2}_{\mathcal{N}}(\Omega)} > 0$.
Assume the contradiction, then there exists a sequence $\{u_n\} \subset \Lambda \setminus \{0\}$ such that $\|u_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)} \to 0$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)}}$. Then $\|v_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)} = 1$ and v_n satisfies

$$1 \le (1+\zeta)\int_\Omega f'(u_n)v_n^2, \quad \forall n. \tag{3.3}$$

Since $u_n \to 0$ in $W^{m,2}_{\mathcal{N}}(\Omega)$, by Adams' embedding for the higher order derivative in Theorem 1.3 we get $f'(u_n) \to f'(0)$ in $L^r(\Omega)$ for all $r \ge 1$. Since v_n is bounded in $W^{m,2}_{\mathcal{N}}(\Omega)$, v_n has a weak

limit say v in $W^{m,2}_{\mathcal{N}}(\Omega)$. Certainly $\|v\|_{W^{m,2}_{\mathcal{N}}(\Omega)} \leq 1$ and up to a subsequence denote it same as v_n which converges strongly to v in $L^r(\Omega)$ for all $r \geq 1$. Hence from (3.3) we get

$$\int_\Omega |\nabla^m u|^2 \leq 1 \leq (1+\zeta) f'(0) \int_\Omega v^2. \tag{3.4}$$

This gives a contradiction if $p > 0$ in which case $f'(0) = 0$. If $p = 0$, by assumption

$$\int_\Omega |\nabla^m u|^2 \geq \lambda_1 \int_\Omega v^2 > (1+\zeta)\mu \int_\Omega v^2$$

which gives a contradiction to (3.3) since $f'(0) = \mu$. This proves Step 1.
It is easy to check that using Step 1 and the definition of Λ:

$$\inf_{u \in \Lambda \setminus \{0\}} \int_\Omega (p+2u^2)|u|^{p+2} e^{u^2} = C_0 > 0. \tag{3.5}$$

Step 2: Finally we have,

$$\begin{aligned}
\lambda \left| \int_\Omega hu \right| &\leq \lambda \|u\|_{L^2(\Omega)} \leq \lambda |\Omega|^{\frac{p+2}{2p+8}} \left(\int_\Omega |u|^{p+4} \right)^{\frac{1}{p+4}} \\
&\leq \frac{\lambda |\Omega|^{\frac{p+2}{2p+8}}}{\mu (p+2u^2)|u|^{p+2} e^{u^2})^{\frac{p+3}{p+4}}} (\mu \int_\Omega (p+2u^2)|u|^{p+2} e^{u^2}) \\
&\leq \left(\frac{\lambda |\Omega|^{\frac{p+2}{2p+8}}}{\mu C_0^{\frac{p+3}{p+4}}} \right) (\mu \int_\Omega (p+2u^2)|u|^{p+2} e^{u^2}).
\end{aligned}$$

Hence from the above inequality together with (3.2) and (3.5) the proof is complete. □

Lemma 3.1. *Suppose $\lambda > 0$ be such that (3.1) holds. Then $\mathcal{M}^0 = \{0\}$.*

Proof. Let $u \in \mathcal{M}^0, u \neq 0$. Then we have

$$\int_\Omega |\nabla^m u|^2 = \int_\Omega f(u)u + \lambda \int_\Omega hu, \tag{3.6}$$

$$\int_\Omega |\nabla^m u|^2 = \int_\Omega f'(u)u^2. \tag{3.7}$$

We note that from (3.7)

$$\int_\Omega |\nabla^m u|^2 = \int_\Omega f'(u)u^2 < (1+\zeta) \int_\Omega f'(u)u^2$$

it implies that $u \in \Lambda \setminus \{0\}$. From these two expressions we get

$$\lambda \int_\Omega hu = \int_\Omega (f'(u)u - f(u))u = \mu \int_\Omega (p+2u^2)|u|^{p+2} e^{u^2}$$

which violates the condition (3.1). Therefore $\mathcal{M}^0 = \{0\}$. □

Next we are going to discuss the topological properties of $\mathcal{M}^+$ and $\mathcal{M}^-$. Given $u \in W^{m,2}_{\mathcal{N}}(\Omega) \setminus \{0\}$, we define $\xi_u : \mathbb{R}^+ \to \mathbb{R}$ by

$$\xi_u(s) = s \int_\Omega |\nabla^m u|^2 - \int_\Omega f(su)u. \tag{3.8}$$

The choice of the above function is consequence of the following expression,

$$\langle J'(su), su \rangle = s \left(s \int_\Omega |\nabla^m u|^2 - \int_\Omega f(su)u - \lambda \int_\Omega hu \right).$$

So, $\xi_u(s) = \lambda \int_\Omega hu$ if and only if $su \in \mathcal{M}$, for $s > 0$.
Now we are ready to state the following lemma.

Lemma 3.2. *For every $u \in W^{m,2}_{\mathcal{N}}(\Omega) \setminus \{0\}$ there exists a unique $s_* = s_*(u) > 0$ such that $\xi_u(.)$ has its maximum at s_* with $\xi_u(s_*) > 0$. Also there holds $s_* u \in \Lambda \setminus \{0\}$.*

Proof. Differentiating (3.8) we have,

$$\xi_u'(s) = \int_\Omega |\nabla^m u|^2 - \int_\Omega f'(su)u^2. \tag{3.9}$$

Observe that,

$$\begin{aligned} s^2\xi_u'(s) &= \int_\Omega |\nabla^m (su)|^2 - \int_\Omega f'(su)(su)^2 \\ &= \langle J''(su)su, su\rangle. \end{aligned} \tag{3.10}$$

Now we note that, $\xi_u(.)$ is strictly concave function on $\mathbb{R}^+$, since

$$\xi_u''(s) = -\int_\Omega f''(su)u^3 < 0. \tag{3.11}$$

Also from the range of μ we get

$$\lim_{s\to 0+} \xi_u'(s) > 0 \text{ and}$$
$$\lim_{s\to\infty} \xi_u(s) = -\infty.$$

Hence there exists a unique maximum point of $\xi_u(.)$, say $s_* = s_*(u) > 0$. Now using (3.9) at $s = s_*$ in the definition of ξ_u, we deduce,

$$\begin{aligned} \xi_u(s_*) &= s_*\int_\Omega f'(s_*u)u^2 - \int_\Omega f(s_*u)u, \quad \text{since } \xi_u'(s_*) = 0 \\ &= \frac{1}{s_*}\int_\Omega (f'(s_*u)s_*u - f(s_*u))s_*u \\ &= \frac{\mu}{s_*}\int_\Omega (p + 2(s_*u)^2)|s_*u|^{p+2}e^{(s_*u)^2} > 0. \end{aligned} \tag{3.12}$$

here we note that $f'(s)s - f(s) = \mu(p + 2s^2)|s|^p s e^{s^2}$. Finally

$$s_*\xi_u'(s_*) = \int_\Omega |\nabla^m(s_*u)|^2 - \int_\Omega f'(s_*u)(s_*u)^2 = 0$$

which implies that $s_*u \in \Lambda \setminus \{0\}$. $\square$

Lemma 3.3. *Let λ be such that (3.1) holds. Then, for every $u \in W^{m,2}_{\mathcal{N}}(\Omega) \setminus \{0\}$, there exists a unique $s_=s_-(u) > 0$ such that $s_-u \in \mathcal{M}^-$, $s_- > s_*$ and $J(s_-u) = \max_{s\geq s_*} J(su)\ \ \forall s \in [s_*,\infty), s \neq s_-$. Also if we assume $\int_\Omega hu > 0$, then there exists a unique $s_+ = s_+(u) > 0$ such that $s_+u \in \mathcal{M}^+$. In particular $s_+ < s_*$ and $J(s_+u) \leq J(su)$ for all $s \in [0, s_-]$.*

Proof. Define the functional $\rho_u : [0,\infty) \to \mathbb{R}$ by $\rho_u(s) = J(su)$. Then it is easy to verify that $\rho_u \in C^2([0,\infty], \mathbb{R}) \cap C((0,\infty), \mathbb{R})$. Then we have

$$\rho_u'(s) = \xi_u(s) - \lambda\int_\Omega hu, \ \ \rho_u''(s) = \xi_u'(s), \ \forall t > 0.$$

Now from (3.1) and (3.12) we have,

$$\xi_u(s_*) - \lambda\int_\Omega hu = \frac{1}{s_*}\left\{\mu\int_\Omega (p + 2(s_*u)^2)|s_*u|^{p+2}e^{(s_*u)^2} - \lambda\int_\Omega h(s_*u)\right\} > 0$$

Since $\xi_u(.)$ is strictly decreasing in (s_*,∞) and $\lim_{t\to\infty} \xi_u(s) = -\infty$, there exists a unique $s_- = s_-(u) > s_*$ such that $\xi_u(s_-) = \lambda\int_\Omega hu$. That is $s_-u \in \mathcal{M}$. One has $s_+ > s_*$ and $\rho_u'(s_)) < 0$, we get $s_-u \in \mathcal{M}^-$.

On the other hand when $\int_\Omega hu > 0$ we have $\lim_{s\to 0+} \xi_u(s) < 0$ and which gives for s close to 0, $\xi_u(s) - \lambda\int_\Omega hu < 0$. Hence there exists a unique s_+ such that $\xi_u(s_+) = \lambda\int_\Omega hu$ which implies $s_+u \in \mathcal{M}$. From the graph we see that $\xi_u(.)$ is strictly decreasing in $(0, s_*)$. Hence we have $s_+u \in \mathcal{M}^+$.

And the remaining properties of s_-, s_+ can be proved by analyzing the identity $\rho_u(s) = \xi_u(s) - \lambda\int_\Omega hu$. $\square$

Remark 3.1. *If we define the positive cone* $\mathcal{P} = \{u \in W^{m,2}_{\mathcal{N}}(\Omega) : \int_\Omega hu > 0\}$ *in* $W^{m,2}_{\mathcal{N}}(\Omega)$. *Then we note that* $\mathcal{M}^+ \subset \mathcal{P}$.

The next corollary shows some topological properties of $\mathcal{M}^+, \mathcal{M}^-$.

Corollary 3.1. *Let* $S_{W^{m,2}_{\mathcal{N}}(\Omega)} = \{u \in W^{m,2}_{\mathcal{N}}(\Omega) : \|u\|_{W^{m,2}_{\mathcal{N}}(\Omega)} = 1\}$. *Then there exists a diffeomorphism* $S^+ : S_{W^{m,2}_{\mathcal{N}}(\Omega)} \to \mathcal{M}^-$ *defined by* $S^+(u) = s_+(u)u$. *Also* $\mathcal{M}^+$ *is homeomorphic to* $S_{W^{m,2}_{\mathcal{N}}(\Omega)} \cap \mathcal{P}$.

Proof. The function S^+ is continuous because s_+ is continuous as an application of implicit function theorem applied to $(s,u) \to \xi_u(s) - \lambda \int_\Omega hu$. And we deduce the continuity of $(S^+)^{-1}$ by the fact that $(S^+)^{-1}(w) = \frac{w}{\|w\|}$. In a similar manner we can prove that $\mathcal{M}^+$ is homeomorphic to $S_{W^{m,2}_{\mathcal{N}}(\Omega)} \cap \mathcal{P}$. □

Relying on the embedding of $W^{m,2}_{\mathcal{N}}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q < \infty$ and using the estimate $F(s) \le \frac{\mu|s|^p}{2}(e^{s^2} - 1)$, for all $s \in \mathbb{R}$ we have the following lemma on the lower bound and upper bound.

Lemma 3.4. *There exists* $C_1, C_2 > 0$ *such that*

$$-C_2\lambda^{2p+8} \ge \theta_0 \ge -C_1\lambda^{\frac{p+3}{p+4}}.$$

Where, $\theta_0 = \inf\{J(u) : u \in \mathcal{M}\}$.

Proof. We prove the case of the lower bound.
Let $u \in \mathcal{M}$ then from the definition,

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla^m u|^2 - \int_\Omega F(u) - \lambda \int_\Omega hu \\ &= \int_\Omega \left[\frac{1}{2} f(u)u - F(u)\right] - \frac{\lambda}{2}\int_\Omega hu. \end{aligned}$$

We note that a simple calculation gives

$$F(t) \le \frac{\mu|t|^p}{2}(e^{t^2} - 1), \qquad \text{for all } s \in \mathbb{R}. \tag{3.13}$$

Using (3.13) we get

$$\begin{aligned} J(u) &\ge \frac{\mu}{2}\int_\Omega \Big((u^2-1)e^{u^2} + 1\Big) - \frac{\lambda}{2}\int_\Omega hu \\ &\ge \frac{c\mu}{2}\int_\Omega |u|^{p+4} - \frac{\lambda}{2}\int_\Omega hu, \end{aligned} \tag{3.14}$$

since $(s^2-1)e^{s^2} + 1 \ge cs^4$ for some $c > 0$, for all $s \in \mathbb{R}$. By an application of Holder inequality we get

$$\int_\Omega hu \le |\Omega|^{\frac{p+2}{2(p+4)}} \|u\|_{L^{p+4}(\Omega)}. \tag{3.15}$$

From (3.14) and (3.15) we get,

$$J(u) \ge \frac{c\mu}{2}\|u\|^{p+4}_{L^{p+4}(\Omega)} - \left(\frac{\lambda|\Omega|^{\frac{p+2}{2(p+4)}}}{2}\right)\|u\|_{L^{p+4}(\Omega)}. \tag{3.16}$$

By considering the global minimum of the function

$$\omega(x) = \left(\frac{c\mu}{2}\right)x^{p+4} - \left(\frac{\lambda|\Omega|^{\frac{p+2}{2(p+4)}}}{2}\right)x,$$

It can be shown that

$$J(u) \ge -C_1\lambda^{\frac{p+4}{p+3}}.$$

In a similar fashion we can prove the upper bound for J. □

As a consequence of Lemma 3.1 we have:

Lemma 3.5. *Let λ and h satisfy (3.1). Given $u \in \mathcal{M} \setminus \{0\}$ there exists $\delta > 0$ and a differentiable function $s : \{w \in W^{m,2}_{\mathcal{N}}(\Omega) : \|w\|_{W^{m,2}_{\mathcal{N}}(\Omega)} < \delta\} \to \mathbb{R}$, with*

$$s(0) = 1, s(w)(u - w) \in \mathcal{M}, \ \forall \quad \|w\|_{W^{m,2}_{\mathcal{N}}(\Omega)} < \delta$$

and

$$\langle s'(0), v\rangle = \frac{2\int_\Omega \nabla^m u \cdot \nabla^m v - \int_\Omega (f'(u)u + f(u))v - \lambda \int_\Omega hv}{\int_\Omega |\nabla^m u|^2 - \int_\Omega f'(u)u^2} \tag{3.17}$$

Proof. We define the function $G : \mathbb{R} \times W^{m,2}_{\mathcal{N}}(\Omega) \to \mathbb{R}$ by,

$$G(s, w) = s \int_\Omega |\nabla^m (u - w)|^2 - \int_\Omega f(s(u - w))(u - w) - \lambda \int_\Omega h(u - w).$$

Then $G \in C^1(\mathbb{R} \times W^{m,2}_{\mathcal{N}}(\Omega); \mathbb{R})$ and since $u \in \mathcal{M}$ it implies $G(1, 0) = \int_\Omega |\nabla^m u|^2 - \int_\Omega f(u)u - \lambda \int_\Omega hu = 0$. Also $G_s(1, 0) \neq 0$, indeed $G_s(1, 0) = \int_\Omega |\nabla^m u|^2 - \int_\Omega f'(u)u^2 \neq 0$ thanks to Lemma 3.1. Then by the Implicit Function Theorem, there exists $\delta > 0, s : \{w \in W^{m,2}_{\mathcal{N}}(\Omega) : \|w\| < \delta\} \to \mathbb{R}$ of class C^1 that satisfies:

$$\begin{aligned} G(s(w), w) = 0 &\text{ for all } w \in W^{m,2}_{\mathcal{N}}(\Omega), \|w\|_{W^{m,2}_{\mathcal{N}}(\Omega)} < \delta, \\ s(0) = 1&. \end{aligned}$$

Also

$$\begin{aligned} 0 &= s(w)G(s(w), w) \\ &= \int_\Omega (s(w)|\nabla^m (u - w)|)^2 - \int_\Omega f(s(w)(u - w)(s(w)(u - w)) - \lambda \int_\Omega h(s(w)(u - w)), \end{aligned}$$

that is $s(w)(u - w) \in \mathcal{M}$ for all $w \in W^{m,2}_{\mathcal{N}}(\Omega)$ with $\|w\| < \delta$. Now if we differentiate the identity $G(s(w), w) = 0$ with respect to w, we get

$$0 = \langle G_s(s(w), w) + G_w(s(w), w), v\rangle \text{ for all } v \in W^{m,2}_{\mathcal{N}}(\Omega).$$

Putting $w = 0$ in the above identity

$$0 = \langle G_s(1, 0)s'(0) + G_w(1, 0), v\rangle = G_s(1, 0)\langle s'(0), v\rangle + \langle G_w(1, 0), v\rangle$$

and we deduce from above

$$\begin{aligned} \langle s'(0), v\rangle &= -\frac{\langle G_w(1, 0), v\rangle}{G_s(1, 0)} \\ &= \frac{2\int_\Omega \nabla^m u \cdot \nabla^m v - \int_\Omega (f'(u)u + f(u))v - \lambda \int_\Omega hv}{\int_\Omega |\nabla^m u|^2 - \int_\Omega f'(u)u^2}. \end{aligned}$$

□

4. Local Minimum of J in $W^{m,2}_{\mathcal{N}}(\Omega)$

We are now in a situation to prove the existence of a minimizer for J and hence we guarantee the existence of first solution.

Since $\mathcal{M}$ is a closed set of $W^{m,2}_{\mathcal{N}}(\Omega)$, hence a complete metric space. Now J is bounded below on $\mathcal{M}$. By the Ekeland's Variational Principle there exists a sequence $\{u_n\} \subset \mathcal{M} \setminus \{0\}$ satisfying:

$$J(u_n) < \theta_0 + \frac{1}{n}, \ \ J(v) \geq J(u_n) - \frac{1}{n}\|v - u_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)} \ \forall v \in \mathcal{M} \tag{4.1}$$

Proposition 4.1. *Let λ and h satisfy (3.1). Then*

$$\lim_{n\to\infty} \|J'(u_n)\|_{(W^{m,2}_{\mathcal{N}}(\Omega))^{-1}} = 0.$$

Proof. We proceed in a few steps. With the help of Lemma 3.4 we've $\underline{\lim}_{n\to\infty}\|u_n\|_{W^{m,2}_{\mathcal{N}}} > 0$.
Claim 1: $\underline{\lim}_{n\to\infty}\int_\Omega (p+2u_n^2)|u_n|^{p+2}eu_n^2 > 0$.
If possible let's assume that for a subsequence of $\{u_n\}$, which is still denoted by $\{u_n\}$, we have

$$\lim_{n\to\infty}(p+2u_n^2)|u_n|^{p+2}e^{u_n^2} \to 0 \text{ as } n\to\infty \tag{4.2}$$

Here we note that $u_n \to 0$ in $L^q(\Omega)$ for all $q\in[1,\infty)$ using (4.2), and if $p>0$,

$$\int_\Omega f(u_n)u_n = \mu\int_\Omega |u_n|^{p+2}e^{u_n^2} \to 0 \text{ as } n\to\infty.$$

Therefore we have $\int_\Omega f(u_n)u_n \to 0$, $\int_\Omega hu_n \to 0$ as $n\to\infty$. Which imply that $\|u_n\|_{W^{m,2}_{\mathcal{N}}} \to 0$ as $n\to\infty$ because $\{u_n\}\subset\mathcal{M}$ hence a contradiction to the fact that $\underline{\lim}_{n\to\infty}\|u_n\|_{W^{m,2}_{\mathcal{N}}} > 0$. Similar argument for $p=0$.
Claim 2: $\underline{\lim}_{n\to\infty}\{|\int_\Omega|\nabla^m u|^2 - \int_\Omega f'(u_n)u_n^2| > 0\}$.
Let the claim doesn't hold. Then for a subsequence $\{u_n\}$ we have

$$\int_\Omega|\nabla^m u|^2 - \int_\Omega f'(u_n)u_n^2 = o_n(1).$$

From this and the fact $\underline{\lim}_{n\to\infty}\|u_n\|_{W^{m,2}_{\mathcal{N}}} > 0$ we deduce that,

$$\underline{\lim}_{n\to\infty} f'(u_n)u_n^2 > 0.$$

Therefore we have $u_n\in\Lambda\setminus\{0\}$ for large n. Since $\{u_n\}\subset\mathcal{M}$ we get

$$\begin{aligned} o_n(1) &= \lambda\int_\Omega hu_n + \int_\Omega[f(u_n)-f'(u_n)u_n]u_n \\ &= -\mu\int_\Omega(p+2u_n^2)|u_n|^{p+2}e^{u_n^2} + \lambda\int_\Omega hu_n, \end{aligned}$$

which contradicts (3.1). This completes the proof of the claim.
Now we proof the theorem. Let's assume $\|J'(u_n)\|_{(W^{m,2}_{\mathcal{N}})^{-1}} > 0$ for all large n (otherwise obvious). Now we define $u=u_n\in\mathcal{M}$ and $w=\delta\frac{J'(u_n)}{\|J'(u_n)\|}$ (by Riesz representation theorem, we identify $J'(u_n)$ as an element in $W^{m,2}_{\mathcal{N}}(\Omega)$ still denote $J'(u_n)$) for $\delta>0$ small. Therefore we can apply Lemma 3.5 for w small we get $s_n(\delta) := s\left[\delta\frac{J'(u_n)}{\|J'(u_n)\|}\right] > 0$ such that

$$w_\delta = s_n(\delta)\left[u_n - \delta\frac{J'(u_n)}{\|J'(u_n)\|}\right]\in\mathcal{M}.$$

Now from (4.1) and a Taylor expansion we have:

$$\begin{aligned} \frac{1}{n}\|w_\delta - u_n\| &\geq J(u_n) - J(w_\delta) \\ &= (1-s_n(\delta))\langle J'(w_\delta), u_n\rangle + \delta s_n(\delta)\left\langle J'(w_\delta), \frac{J'(u_n)}{\|J'(u_n)\|}\right\rangle + o(\delta) \end{aligned}$$

Dividing by $\delta>0$ and taking limit as $\delta\to0$ we get:

$$\frac{1}{n}(1+|s_n'(0)|\|u_n\|) \leq -s_n(0)\langle J'(u_n), u_n\rangle + \|J'(u_n)\| = \|J'(u_n)\|.$$

Hence

$$\|J'(u_n)\| \leq \frac{1}{n}(1+s_n'(0)|\|u_n\|).$$

We complete the proof by using, $|s_n'(0)|$ is uniformly bounded on n by (3.17) and using the Claim 2. ∎

Theorem 4.2. *Let λ, h satisfy (3.1). Then there exists a nonnegative function $u_0\in\mathcal{M}^+$ such that $J(u_0) = \inf_{u\in\mathcal{M}\setminus\{0\}} J(u)$. Moreover, u_0 is a local minimum for J in $W^{m,2}_{\mathcal{N}}(\Omega)$.*

Proof. Let $\{u_n\}$ be a sequence which minimizes J on $\mathcal{M}\setminus\{0\}$ as in (4.1).
Step 1: $\liminf_{n\to\infty}\int_\Omega hu_n>0$ and hence $u_n\in\mathcal{M}^+$. Indeed $u_n\in\mathcal{M}$ and making some suitable adjustments

$$\begin{aligned} J(u_n)=\frac{p}{2(p+2)}\int_\Omega|\nabla^m u|^2+\int_\Omega\left(\frac{1}{p+2}f(u_n)u_n-F(u_n)\right)\\ -\lambda\frac{p+1}{p+2}\int_\Omega hu_n<-C\lambda^{2p+8}. \end{aligned}\tag{4.3}$$

Thanks to Lemma 3.4 there exists $C>0$. Now we note that $F(t)<\frac{1}{p+2}f(t)t$ for all $t\in\mathbb{R}$. Therefore we've from (4.3), to make the inequality consistent with sign that

$$\liminf_{n\to\infty}\int_\Omega hu_n>0.$$

Step 2: $\limsup_{n\to\infty}\|u\|_{W^{m,2}_{\mathcal{N}}(\Omega)}<\infty$.
Case 1. If $p>0$ then by the means of Sobolev embedding we derive boundedness of $\{u_n\}$ in $W^{m,2}_{\mathcal{N}}(\Omega)$. Using the fact from (4.3)

$$\int_\Omega|\nabla^m u|^2\le\lambda\int_\Omega hu_n.$$

Case 2. If $p=0$ by using the fact that $\frac{1}{2}f(t)t-F(t)\ge Ct^4$ for all $t\in\mathbb{R}$ and for some $C>0$ we deduce that $\{u_n\}$ is a bounded sequence in $L^2(\Omega)$. And this gives that $\{F(u_n)\}$ is a bounded sequence in $L^1(\Omega)$ using (4.3) and hence $\{u_n\}$ is a bounded sequence in $W^{m,2}_{\mathcal{N}}(\Omega)$.
Step 3: Existence of $u_0\in\mathcal{M}^+$.
From the previous step up to a subsequence, $u_n\rightharpoonup u_0$ in $W^{m,2}_{\mathcal{N}}(\Omega)$. Now from the Proposition 2.2 we note that $\{f(u_n)u_n\}$ is a bounded sequence in $L^1(\Omega)$. Therefore from Vitali's convergence theorem (for details see Lemma 8.3 in [8]), we get that

$$\int_\Omega f(u_n)\phi\to\int_\Omega f(u_0)\phi,\ \text{for all } \phi\in W^{m,2}_{\mathcal{N}}(\Omega).$$

Hence u_0 will solve (P), in particular $u_0\in\mathcal{M}$. Here we note that $u_0\neq0$ as $h\neq0$ that is $u_0\in\mathcal{M}\setminus\{0\}$. We see that $\theta_0\le J(u_0)$. From (4.3) we get by using Fatou's Lemma that $\theta_0=\liminf_{n\to\infty}J(u_n)\ge J(u_0)$. Therefore u_0 minimizes J on $\mathcal{M}\setminus\{0\}$. Now we have to show $u_0\in\mathcal{M}^+$. From the existence of $s_-(u_0)$ and $s_+(u_0)$ in Lemma 3.3 and using the fact $J(s_+(u_0)u_0)<J(s_-(u_0)u_0)$ we get $u_0\in\mathcal{M}^+$.
Step 4: u_0 is a local minimum for for J in $W^{m,2}_{\mathcal{N}}(\Omega)$.

We see that $s_+(u_0)=1$ because $u_0\in\mathcal{M}^+$ from Step 3. Also we have from the (3.3) we have

$$s_+(u_0)=1<s_*(u_0)$$

Now by the continuity of $s_*(u_0)$, for sufficiently small $\delta>0$

$$1<s_*(u_0-w),\ \forall w\in W^{m,2}_{\mathcal{N}}(\Omega),\|w\|_{W^{m,2}_{\mathcal{N}}(\Omega)}<\delta.\tag{4.4}$$

Now by the Lemma 3.5 for $\delta>0$ small enough if necessary, let $s:\{w\in W^{m,2}_{\mathcal{N}}(\Omega):\|w\|<\delta\}\to\mathbb{R}$ such that $s(w)(u_0-w)\in\mathcal{M}$ and $s(0)=1$. Whenever $s(w)\to1$ when $\|w\|\to0$, we can assume that

$$s(w)<s_*(u_0-w),\ \forall w\in W^{m,2}_{\mathcal{N}}(\Omega),\|w\|<\delta.$$

Hence we get $s(w)(u_0-w)\in\mathcal{M}^+$ using the above inequality and Lemma 3.3. Again by using the Lemma 3.3 we see,

$$J(s(u_0-w)\ge J(s(w)(u_0-w))\ge J(u_0),\ \forall s\in[0,s_*(u_0-w)].$$

Hence from (4.4) we observe that $J(u_0-w)\ge J(u_0)$ for every $\|w\|_{W^{m,2}_{\mathcal{N}}(\Omega)}<\delta$. This shows that u_0 is a local minimizer.
Step 5: A positive local minimum for J. If $u_0\ge0$ then we get the positivity by using the strong maximum principle. In case if $u_0\ngeq0$ then we consider $\tilde{u}_0=s_+(u_0)|u_0|>0\in\mathcal{M}^+$ and also from the definition $\rho_{u_0}(s)=\rho_{|u_0|}(s)$ for all $s>0$. Therefore we get $s_*(|u_0|)=s_*(u_0)$ and from the

definition of s_+ we deduce $s_+(u_0) \leq s_+(|u_0|)$. Hence from Step 4, $s_+(|u_0|) \geq 1$. Therefore by Lemma 3.3 we get $J(\tilde{u}_0) \leq J(|u_0|)$. Now using the assumption $h \geq 0$ in Ω, we have $J(|u_0|) \leq J(u_0)$ and which implies that $\tilde{u}_0$ minimizes J on $\mathcal{M} \setminus \{0\}$. Hence by repeating the same argument as in Step 4 we get the desired result. □

5. Existence of The Second Solution

The existence of the second solution for (P) depends on whether we can apply some version of Mountain Pass Lemma. We wish to look for a solution of the form $u_1 = v + u_0$ where u_0 is the local minimum for the functional (2.1). Then we see that u_1 will solve (P) whenever v solves the following equation:

$$(P_1) \qquad \left\{ \begin{array}{ll} \left.\begin{array}{rl} (-\Delta)^m v & = f(v+u_0) - f(u_0) \\ v & > 0 \end{array}\right\} & \text{in } \Omega, \\ v = \Delta v = 0 = .. = \Delta^{m-1} u & \text{on } \partial\Omega. \end{array} \right.$$

We can write the above PDE as following

$$(\tilde{P}) \qquad \left\{ \begin{array}{l} \left.\begin{array}{rl} (-\Delta)^m 2v & = \tilde{f}(x,v) \\ v & > 0 \end{array}\right\} \text{ in } \Omega, \\ v = \Delta v = 0 = .. = \Delta^{m-1} v \text{ on } \partial\Omega, \end{array} \right.$$

by introducing the function $\tilde{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ and we define by

$$\begin{aligned} \tilde{f}(x,s) &= f(s+u_0(x)) - f(u_0(x)) \text{ if } s \geq 0, \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

The energy functional corresponding to $(\tilde{P})$ is $J_{u_0} : W^{m,2}_{\mathcal{N}}(\Omega) \to \mathbb{R}$ defined by

$$J_{u_0}(v) = \frac{1}{2}\int_\Omega |\nabla^m v|^2 - \int_\Omega \tilde{F}(x,v)dx,$$

where $\tilde{F}(x,s) = \int_0^s \tilde{f}(x,t)dt$. Now onwards, we denote J_{u_0} by J_0. These type of functionals were studied by [12], [2]. We now state the Generalized Mountain Pass Lemma that was introduced by Ghoussoub-Preiss [3].

Definition 5.1. *Let H be a closed subspace of the Banach Space $W^{m,2}_{\mathcal{N}}(\Omega)$. We say that a sequence $\{v_n\} \subset W^{m,2}_{\mathcal{N}}(\Omega)$ is a Palais-Smale sequence for J_0 at the level c around H if:*

(i) $\lim_{n\to\infty} dist(v_n, H) = 0$
(ii) $\lim_{n\to\infty} J_0(v_n) = c$
(iii) $\lim_{n\to\infty} \|J_0'(v_n)\|_{(W^{m,2}_{\mathcal{N}}(\Omega))^{-1}} = 0.$

And we say such a sequence a $(PS)_{H,c}$ sequence.

Remark 5.1. *In case $H = W^{m,2}_{\mathcal{N}}(\Omega)$, the above definition coincides with the usual Palais-Smale sequence at the level c.*

Lemma 5.1. *Let $H \subset W^{m,2}_{\mathcal{N}}(\Omega)$ be a closed set,$c \in \mathbb{R}$. Assume $\{v_n\} \subset W^{m,2}_{\mathcal{N}}(\Omega)$ be a $(PS)_{H,c}$ sequence. Then (upto a subsequence), $v_n \rightharpoonup v_0$ in $W^{m,2}_{\mathcal{N}}(\Omega)$, and*

$$\lim_{n\to\infty}\int_\Omega \tilde{f}(x,v_n) = \int_\Omega \tilde{f}(x,v_0), \qquad \lim_{n\to\infty}\int_\Omega \tilde{F}(x,v_n) = \int_\Omega \tilde{F}(x,v_0). \tag{5.1}$$

Proof. From the fact that $\{v_n\}$ is a $(PS)_{H,c}$ sequence we have:

$$\frac{1}{2}\int_\Omega |\nabla^m v_n|^2 - \int_\Omega \tilde{F}(x,v_n) = c_0 + o_n(1), \tag{5.2}$$

$$\left|\int_\Omega \nabla^m v_n \cdot \nabla^m \phi - \int_\Omega \tilde{f}(x,v_n)\phi\right| \leq o_n(1)\|\phi\|_{W^{m,2}_{\mathcal{N}}(\Omega)}, \qquad \forall \phi \in W^{m,2}_{\mathcal{N}}(\Omega). \tag{5.3}$$

Now we claim that,
Claim: $\sup_n \|v_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)} < \infty$, $\quad \sup_n \int_\Omega \tilde{f}(x, v_n) < \infty$.
Given any $\epsilon > 0$ there exists $s_\epsilon > 0$ such that

$$\int_\Omega \tilde{F}(x,s) \le \epsilon s \tilde{f}(x,s) \text{ for all } |s| \ge s_\epsilon. \tag{5.4}$$

Using (5.2) and (5.4), we see

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla^m v_n|^2 &\le \int_{\Omega\cap\{|v_n|\le s_\epsilon\}} \tilde{F}(x,v_n) + \int_{\Omega\cap\{|v_n|\ge s_\epsilon\}} \tilde{F}(x,v_n) + c + o_n(1)\\ &\le \int_{\Omega\cap\{|v_n|\le s_\epsilon\}} \tilde{F}(x,v_n) + \epsilon\int_\Omega \tilde{f}(x,v_n)v_n + c + o_n(1)\\ &\le C_\epsilon + \epsilon\int_\Omega \tilde{f}(x,v_n)v_n. \end{aligned}\tag{5.5}$$

Now from (5.5) we obtain,

$$\begin{aligned}\int_\Omega \tilde{f}(x,v_n)v_n &\le \int_\Omega |\nabla^m v_n|^2 + o_n(1)\|v_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)}\\ &\le 2C_\epsilon + 2\epsilon\int_\Omega \tilde{f}(x,v_n)v_n + o_n(1)\|v_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)}\end{aligned}$$

by substituting $\phi = v_n$ in (5.3).
Hence by choosing ϵ small enough if needed we get

$$\int_\Omega \tilde{f}(x,v_n)v_n \le \frac{2C_\epsilon}{1-2\epsilon} + o_n(1)\|v_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)}. \tag{5.6}$$

We conclude the claim using (5.6), (5.3) and also $\sup_n \int_\Omega \tilde{f}(x,v_n)v_n < \infty$.
Since $\{v_n\} \subset W^{m,2}_{\mathcal{N}}(\Omega)$ is bounded, up to a subsequence, $v_n \rightharpoonup v_0$ in $W^{m,2}_{\mathcal{N}}(\Omega)$, for some $v_0 \in W^{m,2}_{\mathcal{N}}(\Omega)$.
To prove (5.1) we consider A to be a $2m$ dimensional Lebesgue measure of a set $A \subset \mathbb{R}^{2m}$.

Let $C = \sup_n \int_\Omega |\tilde{f}(x,v_n)v_n| < \infty$ from the above claim. Given $\epsilon > 0$, we define

$$\mu_\epsilon = \max_{x\in\bar{\Omega}, |s|\le \frac{2C}{\epsilon}} |\tilde{f}(x,s)s|.$$

Then, for any $A \subset \Omega$ with $|A| \le \frac{\epsilon}{2C}$, we have

$$\begin{aligned}\int_A |\tilde{f}(x,v_n)| &\le \int_{A\cap\{|v_n|\ge\frac{2C}{\epsilon}\}} \frac{|\tilde{f}(x,v_n)v_n|}{|v_n|} + \int_{A\cap\{|v_n|\le\frac{2C}{\epsilon}\}} |\tilde{f}(x,v_n)|\\ &\le \frac{\epsilon}{2} + \mu_\epsilon|A| \le \epsilon.\end{aligned}$$

Hence $\{\tilde{f}(x,v_n)\}$ is an equi-integrable family in $L^1(\Omega)$ and so is $\{\tilde{F}(x,v_n)\}$(we note that $|\tilde{F}(x,t)| \le C_1|\tilde{f}(x,t)|$ for all $x \in \bar{\Omega}, t \in \mathbb{R}$, for some $C_1 > 0$). By applying the Vitali's convergence theorem we get conclude the lemma. □

Certainly $J_0(0) = 0$ and $v = 0$ is a local minimum for J_0. Also we have

$$\lim_{s\to\infty} J_0(sv) = -\infty \text{ for any } v \in W^{m,2}_{\mathcal{N}}(\Omega) \setminus \{0\}.$$

Hence we can fix $e \in W^{m,2}_{\mathcal{N}}(\Omega) \setminus \{0\}$ such that $J_0(e) < 0$. Now we define the mountain pass level

$$c_0 = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} J_0(\gamma(s)).$$

Where $\Gamma = \{\gamma \in C([0,1], W^{m,2}_{\mathcal{N}}(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}$. Then from the definition of c_0 it follows $c_0 \ge 0$. Define $R_0 = \|e\|_{W^{m,2}_{\mathcal{N}}(\Omega)}$, we note that $\inf\{J_0(v) : \|v\|_{W^{m,2}_{\mathcal{N}}(\Omega)} = R\} = 0$ for all $R \in (0, R_0)$. And we now let $H = W^{m,2}_{\mathcal{N}}(\Omega)$ if $c_0 > 0$ and $H = \{\|v\|_{W^{m,2}_{\mathcal{N}}(\Omega)} = \frac{R_0}{2}\}$ if $c_0 = 0$. We now state as now the lemma giving upper bound for c_0

Lemma 5.2. *The upper bound of the Mountain Pass level is below*

$$c_0 < \frac{(4\pi)^m m!}{2} = \frac{\beta_{2m,m}}{2}. \tag{5.7}$$

Proof. Without loss of generality we can assume that the unit ball $B_0(1) \subset \Omega$. For any $\epsilon > 0$ we define

$$\tilde{\tau}_n(x) := \begin{cases} \sqrt{\frac{1}{2M}\log n} + \frac{1}{\sqrt{2M\log n}}\sum_{\gamma=1}^{m-1}\frac{(1-k|x|^2)^\gamma}{\gamma} & |x| \in [0, \frac{1}{\sqrt{n}}) \\ -\sqrt{\frac{2}{M\log n}}\log|x|, & |x| \in [\frac{1}{\sqrt{n}}, 1), \\ \chi_n(x), & |x| \in [1,\infty) \end{cases} \tag{5.8}$$

where

$$M = \frac{(4\pi)^m(m-1)!}{2}, \qquad \chi_n \in \mathcal{C}_0^\infty(\Omega), \quad \chi_n|_{\partial B_1(0)} = \chi_n|_{\partial\Omega} = 0.$$

Furthermore, for $\gamma = 1, 2, .., m-1$, $D^\gamma \chi_n|_{\partial B_1(0)} = (-1)^\gamma(\gamma-1)!\sqrt{\frac{2}{M\log n}}$, $\Delta^j\chi_n|_{\partial\Omega} = 0$ for $j = 0, 1, 2, ..., [(m-1)/2]$ and $\chi_n, |\nabla\chi_n|, \Delta\chi_n$ are all $O\left(\frac{1}{\sqrt{2\log n}}\right)$. Then, $\tilde{\tau}_n \in W^{m,2}_{\mathcal{N}}(\Omega)$. Now we normalize $\tilde{\tau}_n$, setting

$$\tau_n := \frac{\tilde{\tau}_n}{\|\tilde{\tau}_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)}} \in W^{m,2}_{\mathcal{N}}(\Omega).$$

Suppose (5.7) is not true. This means that, for some $s_n > 0$ (see [4]),

$$J_0(s_n\tau_n) = \sup_{s>0} J_0(s\tau_n) \geq \frac{(4\pi)^m m!}{2} \qquad \forall n.$$

Hence

$$\frac{s_n^2}{2} - \int_\Omega \tilde{F}(x, s_n\tau_n)\frac{(4\pi)^m m!}{2} \qquad \forall n. \tag{5.9}$$

It follows that $\frac{d}{ds}J_0(s\tau_n) = 0$ at the point of maximum $s = s_n$ for J_0, we get

$$s_n^2 = \int_\Omega \tilde{f}(x, s_n\tau_n)(s_n\tau_n). \tag{5.10}$$

Now we note that from the definition of $\tilde{f}$ we see that $\inf_{x\in\bar{\Omega}} \tilde{f}(x,s) \geq e^{s^2}$ for $|s|$ large. Then from (5.9) we get for sufficiently large n

$$\begin{aligned} s_n^2 &\geq \int_{\{|x|\leq\frac{1}{\sqrt{n}}\}} \tilde{f}(x, s_n\tau_n)(s_n\tau_n) \geq \int_{\{|x|\leq\frac{1}{\sqrt{n}}\}} e^{s_n^2\tau_n^2}(s_n\tau_n) \\ &\geq e^{s_n^2\frac{\log n}{2M}}\frac{s_n}{\sqrt{2M}}\sqrt{\log n}\frac{\alpha_{2m}}{n^m} \\ &= \frac{\alpha_{2m}}{\sqrt{2M}}e^{\left(\frac{s_n^2}{2M}-m\right)\log n}s_n(\log n)^{\frac{1}{2}}, \end{aligned} \tag{5.11}$$

where α_{2m} is the volume of the unit ball in $\mathbb{R}^{2m}$. Using the fact $s_n^2 \geq (4\pi)^m m!$ from (5.9) and (5.11) it follows that s_n is bounded and also $s_n^2 \to (4\pi)^m m!$. Also from (5.11) we note

$$s_n \geq \frac{w_{2m}}{\sqrt{2M}}(\log n)^{\frac{1}{2}}, \text{ for all large } n$$

which gives the contradiction. □

We now prove the theorem regarding the existence of second solution.

Theorem 5.1. *Given a local minimum u_0 of J in $W^{m,2}_{\mathcal{N}}(\Omega)$, there exists a point $v_0 \in W^{m,2}_{\mathcal{N}}(\Omega)$ with $v_0 > 0$ in Ω, such that $J_0'(v_0) = 0$.*

Proof. From Lemma 5.7 we have $c_0 \in \left[0, \frac{(4\pi)^m m!}{2}\right)$. Consider $\{v_n\}$ be a Palais-Smale sequence for J_0 at the level c_0 around H (such a $(PS)_{H,c_0}$ sequence exists [3]). Then up to a subsequence $v_n \rightharpoonup v_0$ in $W^{m,2}_{\mathcal{N}}(\Omega)$ for some $v_0 \in W^{m,2}_{\mathcal{N}}(\Omega)$ by Lemma (5.1) and (5.1) holds. We can easily check that v_0 is a solution of $(\tilde{P})$ and therefore a critical point of J_0. It remains to show that v_0 is not a trivial solution.

Case I. $c_0 = 0, v_0 = 0$. We note that $H = \{\|v\|_{W^{m,2}_{\mathcal{N}}(\Omega)} = \frac{R_0}{2}\}$ in this case. As $\{v_n\}$ is a $(PS)_{H,c}$ sequence we have $v_n \to 0$ strongly in $W^{m,2}_{\mathcal{N}}(\Omega)$. From the fact that $dist(v_n, H) = 0$ and H is closed we conclude that $v_n \in H$ and which implies that $v_0 \in H$ and v_0 is different from 0.

Case II. $c_0 \in \left(0, \frac{(4\pi)^m m!}{2}\right), v_0 = 0$. Using the fact that $J_0(v_n) \to c_0$ we see that for given any $\epsilon > 0, \|v_n\|^2_{W^{m,2}_{\mathcal{N}}(\Omega)} \leq (4\pi)^m m! - \epsilon$ for all large n. Let $0 < \delta < \frac{\epsilon}{(4\pi)^m m!}$ and $q = \frac{(4\pi)^m m!}{(1+\delta)((4\pi)^m m! - \epsilon)} > 1$. We have

$$\int_\Omega |\tilde{f}(x, v_n)v_n|^q \leq C \int_\Omega e^{((1+\delta)q\|v_n\|^2)\left(\frac{v_n^2}{\|v_n\|^2}\right)^2},$$

since $\sup_{x\in\bar{\Omega}} |\tilde{f}(x, s)s| \leq Ce^{(1+\delta)s^2}$, for all $s \in \mathbb{R}$, for some $C > 0$. Now from the Tarsi's embedding 1.4 we get that $\sup_{x\in\bar{\Omega}} \int_\Omega |\tilde{f}(x, v_n)v_n|^q < \infty$ since $(1 + \delta)q\|v_n\|^2 \leq (4\pi)^m m!$. Also by Vitali's convergence theorem we get $\int_\Omega \tilde{f}(x, v_n)v_n \to 0$ as $n \to \infty$ since $v_n \to 0$ pointwise almost everywhere in Ω. Which implies

$$\begin{aligned} o_n(1)\|v_n\|_{W^{m,2}_{\mathcal{N}}(\Omega)} &= \langle J_0'(v_n), v_n\rangle = \frac{1}{2}\int_\Omega |\nabla v_n|^2 - \int_{Om} \tilde{f}(x, v_n)v_n \\ &= \frac{1}{2}\int_\Omega |\nabla^m v_n|^2 + o_n(1) \end{aligned}$$

which contradicts the fact $\frac{1}{2}\int_\Omega |\nabla^m v_n|^2 \to c_0$ as $n \to \infty$. Therefore v_0 is not identically 0 in Ω. And positivity of v_0 comes from the fact that $\tilde{f}(x, s) \geq 0$ for all $(x, s) \in \Omega \times \mathbb{R}$ and using the maximum principle. □

6. Proof of Theorem 1.1

Define $\lambda_* = \mu C_0^{\frac{p+3}{p+4}} |\Omega|^{-\frac{p+2}{2p+8}}$ where C_0 is same as in the Proposition (3.1). Then condition (3.1) is true whenever $0 < \lambda < \lambda_*$. From the Theorem 4.2 and 5.1 we show the existence of at least two positive solutions for (P).

Let ϕ_1 be the eigen function of $(-\Delta)^m$ on $W^{m,2}_{\mathcal{N}}(\Omega)$. Define

$$\lambda^* = p\left(\frac{\lambda_1}{p+1}\right)^{\frac{p+1}{p}} \left(\frac{\int_\Omega \phi_1}{\int_\Omega h\phi_1}\right).$$

We prove that there is no solution of (P) when $\lambda > \lambda^*$. Assume that u_λ be a solution of (P). By multiplying ϕ_1 with (P) and performing integration by parts over Ω, we get

$$\int_\Omega (-\Delta)^m u_\lambda \phi_1 = \int_\Omega f(u_\lambda)\phi_1 + \lambda \int_\Omega h\phi_1$$

implies

$$\lambda \int_\Omega h\phi_1 = \int_\Omega (\lambda_1 u_\lambda - f(u_\lambda))\phi_1 \tag{6.1}$$

We see that $\lambda_1 t - f(t) \leq \lambda_1 - \mu t^{p+1} = \Theta(t)$ for all $t > 0$. The global maximum for the function Θ is $p\left(\frac{\lambda_1}{p+1}\right)^{\frac{p+1}{p}}$ on $(0, \infty)$. Then from (6.1) and the definition of λ^* we get $\lambda \leq \lambda^*$. This completes Theorem 1.1.

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TIFR Centre For Applicable Mathematics, Post Bag No. 6503, Shardanagar, Bangalore-560065, Karnataka, India.
E-mail address: `abhishek@math.tifrbng.res.in`